Publ. Mat. **61** (2017), 517–527
DOI: 10.5565/PUBLMAT6121707

A NOTE ON COVERS OF FIBRED HYPERBOLIC MANIFOLDS

Jérôme Los*, Luisa Paoluzzi*, and António Salgueiro†

Abstract: For each surface S of genus $g > 2$ we construct pairs of conjugate pseudo-Anosov maps, φ_1 and φ_2, and two non-equivalent covers $p_i\colon \tilde{S} \longrightarrow S$, $i = 1, 2$, so that the lift of φ_1 to $\tilde{S}$ with respect to p_1 coincides with one of φ_2 with respect to p_2.

The mapping tori of the φ_i and their lift provide examples of pairs of hyperbolic 3-manifolds so that the first is covered by the second in two different ways.

2010 Mathematics Subject Classification: Primary: 57M10; Secondary: 57M50, 57M60, 37E30.

Key words: Regular covers, mapping tori, (pseudo-)Anosov diffeomorphisms.

1. Introduction

Given a finite group G acting freely on a closed orientable surface $\tilde{S}$ of genus larger than 2 one considers the space X of the orbits for the G-action on $\tilde{S}$. The projection $\tilde{S} \longrightarrow X$ is a regular cover and X is again a surface, of genus $g \geq 2$, whose topology is totally determined by the order of G. Assume now that G contains two normal subgroups, H_1 and H_2, non isomorphic but with the same indices in G. In this situation one can construct the following commutative diagram of regular coverings:

$$\begin{tikzcd} & \tilde{S} \arrow[dl] \arrow[dr] & \\ S_1 = \tilde{S}/H_1 \arrow[dr] & & S_2 = \tilde{S}/H_2 \arrow[dl] \\ & X = \tilde{S}/G & \end{tikzcd}$$

*Partially supported by ANR project 12-BS01-0003-01.
†Partially supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

We are interested in the following:

Question. *Is there a pseudo-Anosov diffeomorphism φ of X which lifts to pseudo-Anosov diffeomorphisms φ_1, φ_2, and $\tilde{\varphi}$ of S_1, S_2, and $\tilde{S}$ respectively such that there is a diffeomorphism $g\colon S_1 \longrightarrow S_2$ conjugating φ_1 to φ_2, i.e. $\varphi_2 = g \circ \varphi_1 \circ g^{-1}$?*

The aim of the present note is to provide explicit constructions of surface coverings and pseudo-Anosov diffeomorphisms satisfying the above properties. This will be carried out in the next sections. More explicitly, we prove:

Theorem 1. *For each closed oriented surface S of genus greater than 2, there exists an infinite family of pairs $(\varphi_1, \varphi_2\colon S \longrightarrow S)$ of conjugate pseudo-Anosov maps and two non-equivalent coverings $p_i\colon \tilde{S} \longrightarrow S$ such that a lift of φ_1 with respect to p_1 and a lift of φ_2 with respect to p_2 are the same map $\tilde{\varphi}\colon \tilde{S} \longrightarrow \tilde{S}$.*

Here, the expression *infinitely many pairs of diffeomorphisms* means that there is an infinite family of pairs so that if φ_i and φ'_j belong to different pairs then no power of φ_i is a power of φ'_j, for $i, j = 1, 2$, up to conjugacy. The maps in Theorem 1 come from lifting Anosov diffeomorphisms on a torus to its branched covers.

A positive answer to our initial question implies the existence of hyperbolic 3-manifolds with interesting properties. By considering the mapping tori of the four diffeomorphisms φ, φ_1, φ_2, and $\tilde{\varphi}$, one gets four hyperbolic 3-manifolds N, M_1, M_2, and $\tilde{M}$ respectively. The covers of the surfaces $\tilde{S}$, $\tilde{S}_1$, $\tilde{S}_2$, and X induce covers of these manifolds:

$$\begin{tikzcd} & \tilde{M} \arrow[dl] \arrow[dr] & \\ M_1 \arrow[dr] & & M_2 \arrow[dl] \\ & N & \end{tikzcd}$$

Since φ_1 and φ_2 are conjugate, we see that M_1 and M_2 are homeomorphic (and hence isometric by Mostow's rigidity theorem [**Mo**]). It follows that $\tilde{M}$ is a regular cover of a manifold $M \cong M_1 \cong M_2$ in two different ways.

Corollary 2. *There exists an infinite family of pairs of hyperbolic* 3*-manifolds* $(\tilde{M}, M)$*, such that there exist two non-equivalent regular covers* $p_1, p_2 \colon \tilde{M} \longrightarrow M$ *with non isomorphic covering groups. Moreover, for each* $k \in \mathbb{N}$*, there is a* 3*-manifold* $\tilde{M}$*, which belongs to at least* k *distinct such pairs* $(\tilde{M}, M_\ell)$*,* $1 \leq \ell \leq k$*.*

The existence of hyperbolic 3-manifolds with this type of behaviour was already remarked in [**RS**] but our examples show that one can moreover ask for the manifolds to fibre over the circle and for the two group actions to preserve a fixed fibration (see also Section 3 for other comments on the two types of examples).

2. Main construction

In this section we answer in the positive a weaker version of our original question, where the diffeomorphisms involved are not required to be pseudo-Anosov.

2.1. Symmetric surfaces. For every pair of integers $n, m \geq 1$ we will construct a closed connected orientable surface of genus $nm+1$ admitting a symmetry of type $G = \mathbb{Z}/n \times \mathbb{Z}/m$.

Let n and m be fixed. Consider the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ and the following G-action: the generator of $\mathbb{Z}/n$ is $(x, y) \mapsto (x + 1/n, y)$ and that of $\mathbb{Z}/m$ is $(x, y) \mapsto (x, y + 1/m)$, where all coordinates are thought mod 1.

The union of the sets of lines $L_x = \{(i/n, y) \in \mathbb{R}^2 \mid i \in \mathbb{Z},\ y \in \mathbb{R}\}$ and $L_y = \{(x, j/m) \in \mathbb{R}^2 \mid j \in \mathbb{Z},\ x \in \mathbb{R}\}$ maps to a G-equivariant family $\mathcal{L}$ of simple closed curves of T: n meridians and m longitudes, as in Figure 1.

Consider a standard embedding of T in the 3-sphere $\mathbf{S}^3 \subset \mathbb{C}^2$ so that the G action on the torus is realised by the $(\mathbb{Z}/n \times \mathbb{Z}/m)$-action on $\mathbf{S}^3$ defined as $(z_1, z_2) \mapsto (e^{2i\pi/n} z_1, z_2)$ and $(z_1, z_2) \mapsto (z_1, e^{2i\pi/m} z_2)$. A small G-invariant regular neighbourhood of $\mathcal{L}$ in $\mathbf{S}^3$ is a handlebody $\tilde{\mathcal{H}}$ of genus $nm + 1$. Its boundary is the desired surface $\tilde{S}$.

2.2. The normal subgroups H_1 and H_2.

Notation 1. Let $n \in \mathbb{N}$.

- We denote by $\Pi(n)$ the set of all prime numbers that divide n.
- For any $P \subset \Pi(n)$ we denote by $n_P \in \mathbb{N}$ the divisor of n such that $\Pi(n_P) = P$ and $\Pi(n/n_P) = \Pi(n) \setminus P$.

Definition 1. Let A and B be two finite sets of prime numbers such that

- $A \cap B = \emptyset$;
- $A \cup B \neq \emptyset$.

Let $n, m \in \mathbb{N}$, $n, m \geq 2$. We say that (n, m) is *admissible with respect to* (A, B) if the following conditions are verified:

- $A \cup B \subset \Pi(n) \cap \Pi(m)$;
- $\dfrac{n_A m_B}{m_A n_B}$ is an integer strictly greater than 1, that is m_A divides n_A, n_B divides m_B, and at least one of the divisors is proper.

In this case we let $C = \Pi(n) \setminus (A \cup B)$ and $D = \Pi(m) \setminus (A \cup B)$.

We note that, since $\dfrac{n_A m_B}{m_A n_B}$ is an integer greater than one, then $m_A m_B = m_{A\cup B} \neq n_{A\cup B} = n_A n_B$. This definition of admissibility will be used to guarantee, in the proof of Lemma 3, that there is a prime $p \in A \cup B$ such that the Sylow p-subgroup of H_1 is cyclic but not that of H_2.

Remark 1. If $\gcd(n, m) = d > 1$ and at least one between $\gcd(d, n/d)$ and $\gcd(d, m/d)$ is not 1, then there is a choice of sets A, B such that (n, m) is admissible with respect to (A, B). Note that this choice may not be unique. In fact, for each $k \in \mathbb{N}^*$ there is a pair (n, m) such that one has at least k choices of sets (A, B) for which (n, m) is admissible. Let $p_1, \ldots, p_k$ be k distinct prime numbers and consider $n = p_1^2 \ldots p_k^2$ and $m = p_1 \ldots p_k$ so that $n = m^2$. For each $1 \leq \ell \leq k$ let $A_\ell = \{p_\ell\}$ and $B_\ell = \emptyset$, then for each ℓ the pair (n, m) is admissible with respect to (A_ℓ, B_ℓ).

We consider the $G = \mathbb{Z}/n \times \mathbb{Z}/m$-actions on the torus, where (n, m) is admissible with respect to some choice of (A, B) as in Definition 1. Of course we have $\mathbb{Z}/n \cong \mathbb{Z}/n_A \times \mathbb{Z}/n_B \times \mathbb{Z}/n_C$ and $\mathbb{Z}/m \cong \mathbb{Z}/m_A \times \mathbb{Z}/m_B \times \mathbb{Z}/m_D$.

The two subgroups of G we shall consider are:

$$H_1 = (\mathbb{Z}/n_A \times \mathbb{Z}/n_C) \times (\mathbb{Z}/m_B \times \mathbb{Z}/m_D)$$

and

$$H_2 = (\mathbb{Z}/(n_A/m_A) \times \mathbb{Z}/n_B \times \mathbb{Z}/n_C) \times (\mathbb{Z}/m_A \times \mathbb{Z}/(m_B/n_B) \times \mathbb{Z}/m_D)$$

which are obviously normal (since G is abelian) and of the same order:

$$nm/(n_B m_A) = n_A m_B n_C m_D \geq n_A m_B > 1,$$

since the pair (n, m) is admissible with respect to (A, B). Clearly the two subgroups H_1 and H_2 depend on the choice of (A, B).

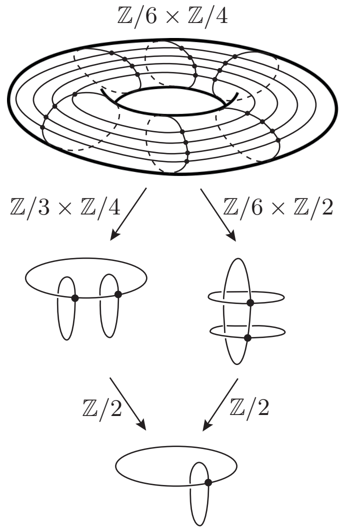

FIGURE 1. The set $\mathcal{L}$ of simple closed curves of T, with 6 meridians and 4 longitudes, and the action of two subgroups $H_1 = \mathbb{Z}/3 \times \mathbb{Z}/4$ and $H_2 = \mathbb{Z}/6 \times \mathbb{Z}/2$ of $G = \mathbb{Z}/6 \times \mathbb{Z}/4$. In this case, $A = \emptyset$, $B = \{2\}$.

Lemma 3. *The two subgroups H_1 and H_2 are not isomorphic but their quotients G/H_1 and G/H_2 are.*

Proof: Since, according to Definition 1, n_A/m_A and m_B/n_B cannot be both equal to 1, there is a prime $p \in A \cup B$ such that the Sylow p-subgroup of H_1 is cyclic but not that of H_2. Finally, we observe that $G/H_1 \cong \mathbb{Z}/n_B \times \mathbb{Z}/m_A \cong \mathbb{Z}/m_A \times \mathbb{Z}/n_B \cong G/H_2$, that is, both quotients are cyclic of order $n_B m_A$, since $A \cap B = \emptyset$. □

2.3. Lifting diffeomorphisms on the different covers. An easy Euler characteristic check shows that $X = \tilde{S}/G$ is a surface of genus 2 bounding a handlebody $\mathcal{H}_X = \tilde{\mathcal{H}}/G$. Similarly, one can verify that $\mathcal{H}_i = \tilde{\mathcal{H}}/H_i$ is a handlebody of genus $n_B m_A + 1$.

We analyse now how the regular coverings $S_i \longrightarrow X$ are built. Consider the following composition of group morphisms

$$\pi_1(X) \longrightarrow \pi_1(\mathcal{H}_X) \longrightarrow H_1(\mathcal{H}_X) \cong \mathbb{Z}^2,$$

where the first map is induced by the inclusion of X as the boundary of $\mathcal{H}_X$. Note that $\pi_1(\mathcal{H}_X)$ is a free group of rank 2 generated by the images μ and λ of a meridian and a longitude of the original torus T. Of course, these two curves can be pushed onto the boundary X of $\mathcal{H}_X$. We can also assume that they have the same basepoint $x_0 \in X$. Let us denote by $[\mu]$ and $[\lambda]$ the classes of μ and λ respectively in $H_1(\mathcal{H}_X)$. There are two natural morphisms from $H_1(\mathcal{H}_X) \cong \mathbb{Z}^2$ to $\mathbb{Z}/n_B m_A \cong \mathbb{Z}/m_A \times \mathbb{Z}/n_B$: the first one maps $[\mu]$ to a generator of $\mathbb{Z}/m_A$ and $[\lambda]$ to a generator of $\mathbb{Z}/n_B$ while the second one exchanges the roles of the two elements and maps $[\mu]$ to a generator of $\mathbb{Z}/n_B$ and $[\lambda]$ to a generator of $\mathbb{Z}/m_A$.

The two coverings $S_i \longrightarrow X$ are determined by the composition of these two group morphisms:

$$\pi_1(X) \longrightarrow \pi_1(\mathcal{H}_X) \longrightarrow H_1(\mathcal{H}_X) \cong \mathbb{Z}^2 \longrightarrow \mathbb{Z}/n_B m_A \cong \mathbb{Z}/m_A \times \mathbb{Z}/n_B$$

that is, the fundamental groups $\pi_1(S_i)$ correspond to the kernels of the two morphisms just constructed.

Lemma 4. *The two coverings $S_i \longrightarrow X$, $i = 1, 2$ are conjugate. More precisely there is a diffeomorphism τ of order 2 of X, inducing a well-defined element $\tau_* \in \mathrm{Aut}(\pi_1(X, x_0))$ such that τ_* exchanges $\pi_1(S_1)$ and $\pi_1(S_2)$.*

Proof: The diffeomorphism τ is the involution with two fixed points, x_0 and y_0 pictured in Figure 2. Note that τ exchanges μ and λ. The fact that τ_* defines an element of $\mathrm{Aut}(\pi_1(X, x_0))$ (and not just $\mathrm{Out}(\pi_1(X, x_0))$) follows from the fact that $\tau(x_0) = x_0$. $\square$

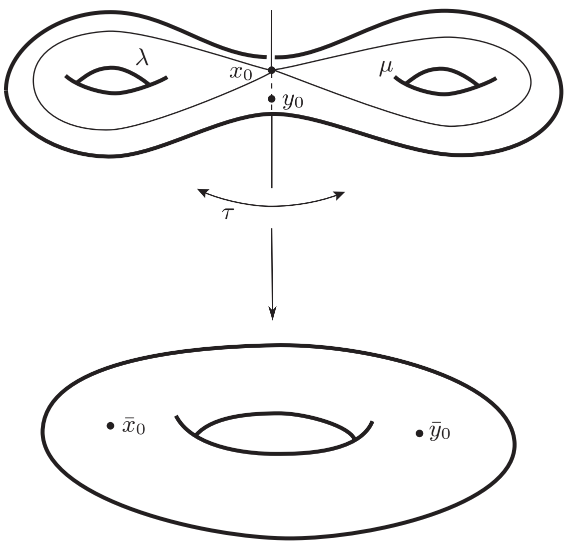

Figure 2. The action of τ on X and the quotient X/τ.

We are interested in diffeomorphisms f of X which commute with τ and fix both x_0 and y_0. We have the following easy fact.

Lemma 5. *A diffeomorphism f of X commutes with τ and fixes both x_0 and y_0 if and only if it is the lift of a diffeomorphism of the torus fixing two points $\bar{x}_0$ and $\bar{y}_0$.*

Proof: Observe that the orbifold quotient X/τ is a torus with two cone points of order 2. Clearly, any diffeomorphism f that commutes with τ and fixes x_0 and y_0 induces a map of X/τ which fixes the two cone points. Vice-versa, given a diffeomorphism of the torus which fixes two points $\bar{x}_0$ and $\bar{y}_0$ we can lift it to X once we choose an identification of the torus with X/τ such that $\bar{x}_0$ and $\bar{y}_0$ are mapped to the two cone points. □

We are interested in diffeomorphisms of X which commute with τ and lift to the covers $S_i \longrightarrow X$, $i = 1, 2$, and $\tilde{S} \longrightarrow X$.

Lemma 6. *Let f be a diffeomorphism of X which commutes with τ and fixes x_0 and y_0. One can choose $k \in \mathbb{N}$ such that f^k lifts to diffeomorphisms of S_1, S_2, and $\tilde{S}$ which fix pointwise the fibres of x_0.*

Proof: The diffeomorphism f fixes x_0 and so induces an automorphism f_* of $\pi_1(X, x_0)$. Choose x_1, x_2, and $\tilde{x}$ points of S_1, S_2, and $\tilde{S}$ respectively which map to x_0. Since $\pi_1(X, x_0)$ is finitely generated, there is a finite number of subgroups of $\pi_1(X, x_0)$ with a given finite index. Since $\pi_1(S_1, x_1)$, $\pi_1(S_2, x_2)$, and $\pi_1(\tilde{S}, \tilde{x})$ have finite index in $\pi_1(X, x_0)$ then there is a power of f_* which leaves $\pi_1(S_1, x_1)$, $\pi_1(S_2, x_2)$, and $\pi_1(\tilde{S}, \tilde{x})$ invariant. As a consequence, the corresponding power of f lifts to S_1, S_2, and $\tilde{S}$. Since each lift acts by leaving the fibre of x_0 invariant, up to possibly passing to a different power, we can assume that the lifts fix pointwise the fibre of x_0. Note moreover that for this to happen it suffices that the fibre of x_0 in the covering $\tilde{S} \longrightarrow X$ is pointwise fixed. □

Remark 2. The argument of the above lemma shows that one can choose a power of f which lifts, as in the statement of the lemma, to any covering of X corresponding to a subgroup K such that $\pi_1(\tilde{S}, \tilde{x}) \subset K \subset \pi_1(X, x_0)$. Recall that each such K is normal in $\pi_1(X, x_0)$, since $G \cong \pi_1(X, x_0)/\pi_1(\tilde{S}, \tilde{x})$ is abelian.

Let f be a diffeomorphism of X commuting with τ and fixing x_0 and y_0, and let φ be a power of f satisfying the conclusions of Lemma 6. Denote by $\tilde{\varphi}$ the lift of φ to $\tilde{S}$ and by φ_1 and φ_2 its projections to S_1 and S_2 respectively. Note that in principle the lift $\tilde{\varphi}$ of φ is not unique:

two possible lifts differ by composition with a deck transformation. In this case, however, since we require that $\tilde{\varphi}$ fixes pointwise the fibre of x_0 while the group G of deck transformations acts freely on it, we can conclude that our choice of $\tilde{\varphi}$ is unique.

Proposition 7. *Let f be a diffeomorphism of X commuting with τ and fixing x_0 and y_0, and let φ be a power of f satisfying the conclusions of Lemma 6. Denote by φ_1 and φ_2 the lifts of φ to S_1 and S_2 respectively, as described above. The maps φ_1 and φ_2 are conjugate.*

Proof: By construction, the involution τ of X lifts to a map g between S_1 and S_2 conjugating a lift of φ on S_1 to a lift of φ on S_2. Since two different lifts differ by composition with a deck transformation, reasoning as in the remark above we see that g conjugates φ_1 to φ_2 since both φ_1 and φ_2 are the only lifts of φ that fix every point in the fibre of x_0. $\square$

3. Proofs of Theorem 1 and Corollary 2, and some remarks on commensurability

In this section we use the construction detailed in Section 2 to prove our main result. We will then discuss some consequences for 3-dimensional manifolds.

3.1. Proof of Theorem 1. By Proposition 7, it is sufficient to show that a pseudo-Anosov $f\colon X \longrightarrow X$ that fixes x_0 and y_0, and commutes with τ, does exist. According to Lemma 5, any such f is the lift of a diffeomorphism $\bar{f}$ of the torus that fixes two points $\bar{x}_0$ and $\bar{y}_0$. Let A be an Anosov diffeomorphism of the torus. Since A has infinitely many periodic orbits (see [**Si**] for instance), we can choose a power $\bar{f}$ of A which fixes two points on the torus. Let f denote the lift of $\bar{f}$ to X. We need to show that f is pseudo-Anosov, that is we need to exclude the possibilities that f is finite order or reducible. The following argument is standard (see [**FLP**, exposé 13]). Clearly f cannot be periodic since its quotient $\bar{f}$ has infinite order. Since, by assumption, $\bar{f}$ is an Anosov map, it admits a pair of invariant foliations $(\mathcal{F}^+, \mathcal{F}^-)$. These lift to invariant foliations $(\tilde{\mathcal{F}}^+, \tilde{\mathcal{F}}^-)$ for f. Note also that x_0 and y_0, which are lifts of the two fixed points of $\bar{f}$, are singular points for the foliations $(\tilde{\mathcal{F}}^+, \tilde{\mathcal{F}}^-)$. If f were reducible then at least one leaf $\tilde{\gamma}$ of $\tilde{\mathcal{F}}^+$ or of $\tilde{\mathcal{F}}^-$ would be fixed by f and connect one singularity between x_0 or y_0 either to itself or to the other one. Such a leaf would project to a leaf of either $\mathcal{F}^+$ or $\mathcal{F}^-$ satisfying the analogous property. This however cannot happen for an Anosov map.

This shows that any f which is the lift of an Anosov map is a pseudo-Anosov map. Any nonzero power φ of a pseudo-Anosov map f is again pseudo-Anosov, and, reasoning as above, so are its lifts φ_1, φ_2, and $\tilde{\varphi}$.

It remains to prove that infinitely many choices of φ_i's do not share common powers. This follows readily from the fact that there exist infinitely many primitive Anosov maps on the torus. □

3.2. Hyperbolic fibred 3-manifolds. The aim of this part is to prove Corollary 2 and compare the examples constructed here to those given in [**RS**].

For each choice of conjugate pseudo-Anosov maps φ_1 and φ_2 and common lift $\tilde{\varphi}$ as in Theorem 1, we can consider the associated mapping tori M_1, M_2, and $\tilde{M}$ respectively. The 3-manifolds thus obtained are hyperbolic according to Thurston's hyperbolization theorem for manifolds that fibre over the circle (see [**O**]). By construction, the mapping tori M_1 of φ_1 and M_2 of φ_2 are homeomorphic, i.e. $M_1 = M_2 = M$ since φ_1 and φ_2 are conjugate. Moreover, again by construction, the mapping torus $\tilde{M}$ of $\tilde{\varphi}$ covers M in two non-equivalent ways.

According to Remarks 1 and 2, for each k one can find pseudo-Anosov maps $\tilde{\varphi}$ which cover at least k pairs of conjugate pseudo-Anosov maps in the fashion described in Theorem 1. This proves the last part of the corollary.

Remark 3. It follows from the construction, that the group G acts on $\tilde{M}$ by isometries which are, moreover, fibration-preserving. In general, one may expect that the isometry group of $\tilde{M}$ is larger than G. Note that if this is the case and if one could find an element $h \in \mathrm{Isom}(\tilde{M})$ which does not normalise G, then the image of the fibration of $\tilde{M}$ by h is another fibration of $\tilde{M}$. This new fibration is not isotopic to the initial one a priori. On the other hand, the conjugate of G by h preserves the new fibration and induces a system of coverings equivalent to the original one.

It remains to show that there are infinitely many pairs of hyperbolic manifolds $(\tilde{M}, M)$ such that the first covers the second in two non-equivalent ways. Note that the fact that Theorem 1 provides infinitely many choices is not sufficient to conclude, since a hyperbolic manifold can admit infinitely many non-equivalent fibrations (see [**Th**]).

The existence of infinitely manifolds follows from the following observation. Up to isomorphism, there are infinitely many groups G to which our construction applies. Each of these groups acts by hyperbolic isometries on some closed $\tilde{M}$. Since the group of isometries of a closed hyperbolic 3-manifold is finite, we can conclude that there are infinitely many pairs of manifolds $(\tilde{M}, M)$ up to hyperbolic isometry and hence, because of Mostow's rigidity theorem [**Mo**], up to homeomorphism.

Another way to reason is the following. Given φ_1, φ_2, and $\tilde{\varphi}$ as above we can consider the mapping tori $M_1^{(k)}$, $M_2^{(k)}$, and $\tilde{M}^{(k)}$ of φ_1^k, φ_2^k, and $\tilde{\varphi}^k$ respectively, for $k \geq 1$. All the manifolds thus obtained are commensurable, and volume considerations show that the manifolds $\tilde{M}^{(k)}$ are pairwise non homeomorphic. Indeed, given a pseudo-Anosov f of X, for any choice of G and of φ_1, φ_2, and $\tilde{\varphi}$, all the mapping tori obtained are commensurable to the mapping torus of f. More precisely all these manifolds are fibred commensurable according to the definition of [**CSW**], that is they admit common fibred covers such that the coverings maps preserve the fixed fibrations.

This latter observation shows that we can construct infinitely many distinct pairs $(\tilde{M}, M)$ such that the first covers the second in two non equivalent ways which are all (fibred) commensurable. Unfortunately we do not know whether the manifolds we construct in Corollary 2 belong to infinitely many distinct commensurability classes as well. A different construction is based on the fact that hyperbolic arithmetic manifolds have a large commensurator [**RS**]. This construction can be made for infinitely many isomorphism classes of quaternion algebras, which shows that it is possible to find infinitely many pairs of manifolds $(\tilde{M}, M)$ such that the first covers the second in two non-equivalent ways and the manifolds $\tilde{M}$ are pairwise non commensurable.

References

[CSW] D. Calegari, H. Sun, and S. Wang, On fibered commensurability, *Pacific J. Math.* **250(2)** (2011), 287–317. DOI: 10.2140/pjm.2011.250.287.

[FLP] A. Fathi, F. Laudenbach, and V. Poénaru, Travaux de Thurston sur les surfaces, *Astérisque* **66–67** (1979), 284 pp.

[Mo] G. D. Mostow, *"Strong Rigidity of Locally Symmetric Spaces"*, Annals of Mathematics Studies **78**, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973.

[O] J.-P. Otal, Le théorème d'hyperbolisation pour les variétés fibrées de dimension 3, *Astérisque* **235** (1996), 159 pp.

[RS] A. Reid and A. Salgueiro, Some remarks on group actions on hyperbolic 3-manifolds, Preprint.

[Si] Ja. G. Sinaĭ, Markov partitions and U-diffeomorphisms, (Russian), *Funkcional. Anal. i Priložen* **2(1)** (1968), 64–89.

[Th] W. P. THURSTON, A norm for the homology of 3-manifolds, *Mem. Amer. Math. Soc.* **59(339)** (1986), i–vi and 99–130.

Jérôme Los and Luisa Paoluzzi:
Aix-Marseille Université, CNRS
Centrale Marseille, I2M, UMR 7373
13453 Marseille
France
E-mail address: jerome.los@univ-amu.fr
E-mail address: luisa.paoluzzi@univ-amu.fr

António Salgueiro:
Department of Mathematics
University of Coimbra
3001-454 Coimbra
Portugal
E-mail address: ams@mat.uc.pt

Primera versió rebuda el 2 de desembre de 2015,
darrera versió rebuda el 23 de gener de 2017.